Analysis of a rotating flow at small Reynolds number

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(Received 18 September 1962 and in revised form 13 November 1963)

A solution of the Navier–Stokes equations is obtained for the flow resulting from the steady rotation of a semi-infinite right circular (solid) cylinder about its vertical axis. Incompressible viscous fluid is assumed to fill the space outside the cylinder on one side of a horizontal solid plane. In the proposed method of solution the pertinent physical quantities are expressed as series in positive powers of the Reynolds number Re with space-dependent coefficients. It is shown that the coefficients of $(Re)^M$ can be obtained by solving linear partial differential equations which depend on the coefficients of $(Re)^i$, where $i < M$. A truncated solution, which holds for small Re, is obtained by solving for the first two coefficients. These results indicate that at the flat end of the cylinder the pressure distribution is nearly constant, yet along the adjacent bounding plane it rises with the radial direction.

1. Introduction

We obtain here the solution for the flow resulting from the steady rotation about its axis of a semi-infinite solid cylinder $z \geqslant d/R$, which is immersed in a large expanse of otherwise quiescent fluid. We let Ω denote its angular velocity and (r, θ, z) cylindrical co-ordinates which are non-dimensionalized with respect to the radius R of the cylinder. The end of the cylinder is parallel to the solid surface $z = 0$, and the space on the one side of this plane and outside the cylinder is completely filled with fluid. It is shown that the dependent variables can be expanded as power series in the Reynolds number $Re(=\Omega R^2/\nu)$, in which the coefficients are the solutions of linear partial differential equations in r and z. The first two of these coefficients are obtained here by relaxation methods.

When u, v, w denote the three components of velocity, ρ the density and p the pressure and when the physical dimensions of the problem are chosen so that ΩR and $\rho\Omega^2R^2$ are unit velocity and unit pressure respectively, the governing equations are

$$Re\left[u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}+\frac{\partial p}{\partial r}\right] = \frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial(ru)}{\partial r}\right\}+\frac{\partial^2 u}{\partial z^2}, \tag{1}$$

$$Re\left[u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}\right] = \frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial(rv)}{\partial r}\right\}+\frac{\partial^2 v}{\partial z^2}, \tag{2}$$

$$Re\left[u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}+\frac{\partial p}{\partial z}\right] = \frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial w}{\partial r}\right\}+\frac{\partial^2 w}{\partial z^2}, \tag{3}$$

$$\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{\partial w}{\partial z} = 0, \tag{4}$$

where all dependent variables are taken to be independent of θ. Kármán (1921), Batchelor (1951), Stewartson (1953), Cochrane (1934) and others have discussed the solutions of equations (1)–(4) which represent the motion of fluids bounded by one or two rotating disks on which there may be suction. Their analyses were based on the assumption that the velocity field has the form

$$u = rf(z), \quad v = rg(z), \quad w = h(z), \tag{5}$$

so that equations (1)–(4) yield a set of ordinary differential equations. Batchelor showed that these can be integrated in the finite or infinite ranges $0 < z < H$ or $0 < z < \infty$ when the boundary conditions are

$$\left.\begin{aligned} u = 0, \quad w = W_a, \quad v = (\omega_a/\Omega)r \quad \text{when } z = 0, \\ u = 0, \quad w = W_h, \quad v = (\omega_h/\Omega)r \quad \text{when } z = H, \end{aligned}\right\} \tag{6a}$$

or

$$u = 0, \quad v = (\omega_\infty/\Omega)r \quad \text{as} \quad z \to \infty. \tag{6b}$$

Here ω_a, ω_h and ω_∞ are the angular velocities of the disks or the liquid at infinity and W_a, W_h are the rates of 'suction' at the surfaces of the disks. It should be pointed out that in these solutions different numbers of boundary conditions are imposed on various parts of the surface bounding the flow: three at $z = 0$ or $z = H$, two at $z = \infty$ and none at $r = \infty$. It is, of course, possible that in the case of the two disks the pressure and velocity distributions at $r = \infty$, $H > z > 0$ do not affect the flow in the vicinity of the origin and hence one should not, or need not, prescribe any conditions there. Nevertheless, the hypothesis that the boundary conditions have only a local effect would not explain why the number of conditions imposed at $z = H$ drops from three to two as H approaches infinity. Nor would it explain why the conditions prevailing at $z = \infty$ have more weight than those prevailing at $r = \infty$. The author believes that these inconsistencies result from the fact that equation (5) is not the most general form of velocity field for either the case of a single disk or the case of two disks.

The incompleteness of the assumed velocity field becomes more apparent when it is treated as a particular case of the form

$$\left.\begin{aligned} u &= \sum_{j=0}^{\infty} f_{2j+1}(z)\,r^{2j+1}, & v &= \sum_{j=0}^{\infty} g_{2j+1}(z)\,r^{2j+1}, \\ w &= \sum_{j=1}^{\infty} h_{2j}(z)\,r^{2j}, & p &= \sum_{j=0}^{\infty} k_{2j}(z)\,r^{2j}. \end{aligned}\right\} \tag{7}$$

Indeed, when this is substituted into (1)–(4) and the first $(2M-1)$ powers of r are compared, $4M$ coupled ordinary differential equations governing the functions f, g, etc., are obtained. With any $M \geqslant 1$ the first four are

$$\begin{aligned} f_1^2 + h_0 f_1' - g_1^2 &= -2k_2 + Re^{-1}(f_1'' + 8f_3), \\ h_0 g_1' + 2f_1 g_1 &= Re^{-1}(g_1'' + 8g_3), \\ h_0 h_0' &= -k_0' + Re^{-1}(h_0'' + 4h_2), \\ 2f_1 + h_0' &= 0. \end{aligned}$$

A sufficient condition for both the integrability of these and the satisfying of all the other $4(M-1)$ equations is that f_3, g_3, h_3 and k_2' as well as all the other 'higher' coefficients should vanish identically. This reduces the $4M$ equations, no matter how large M is, to those four considered by the above-mentioned authors. However, triviality is unlikely to be also the necessary condition for the satisfaction of these $4(M-1)$ equations. Indeed, since with $M > 1$ the differential system given by the $4M$ equations and with the conditions

$$\left.\begin{array}{l} f_{2j+1} = 0 \quad (j \geqslant 0), \quad h_0 = W_a \quad \text{or} \quad W_h, \quad h_{2j} = 0 \quad (j > 0), \\ g_1 = (\omega_a/\Omega) \quad \text{or} \quad (\omega_h/\Omega), \quad g_{2j+1} = 0 \quad (j > 0), \quad \text{when} \quad z = 0, H, \end{array}\right\} \tag{6a}$$

or $$f_{2j+1} = 0 \quad (j \geqslant 0), \quad g_1 = (\omega_\infty/\Omega), \quad g_{2j+1} = 0 \quad (j > 0) \quad \text{as} \quad z \to \infty, \tag{6b}$$

is underdeterminate, and since this underdeterminacy grows with M, it is to be expected that infinitely many other assumptions can be made about the behaviour of the 'higher' coefficient functions. In fact one need not consider the differential system for which $M > 1$, for even with $M = 1$ one imposes an ambiguous requirement, namely, that k_2' should vanish. Thus Kármán and Cochrane take k_2 to be zero while Stewartson assumes that it is a positive constant. We maintain that the extra data required to make the Mth differential system determinate or the choice of k_2 unique depend upon the conditions prevailing at $r = \infty$, and if these conditions are ignored the problem tackled is not well posed.

In view of the incompleteness of the form of the solutions given by (5) and of its inability to consider the effect of the conditions prevailing at $r = \infty$, one may wonder whether the flows which these solutions represent do exist in practice. It is not true that Cochrane's or Stewartson's flow must always take place in the fluid on one side of a large rotating disk (in the vicinity of its centre) or between two parallel disks. These flows can, probably, be produced in a laboratory if the velocity and pressure distributions at 'infinity' in both the analytical solutions and the corresponding experiments are similar. However, Cobb & Saunders's (1956) comparison between the experimental and analytical results as well as Wagner's (1948) analysis seem to imply that Kármán's flow can be reproduced by rotating a disk in a large body of otherwise quiescent viscous fluid. Thus these authors not only overlook the effect of the conditions at $r = \infty$ but also disregard the non-vanishing of Kármán's velocity field as $z \to \infty$. Again, Stewartson discusses his cardboard disks experiment without mentioning the similarity between the physical conditions prevailing at the rim and the corresponding analytical results for $r = \infty$. More serious than these omissions is Popper & Reiner's (1956) erroneous conclusion that the observed phenomenon of negative $\partial p/\partial r$ in the air gap between two disks either contradicts Stewartson's result

$$p = k_0 + k_2 r^2 \quad \text{with} \quad k_2 \geqslant 0,$$

or indicates that in this case the Navier–Stokes equations are inapplicable. It has not yet been shown that at the hole which connects the manometer to their apparatus, as well as at the outer rim of the air gap, the flow resembles Stewartson's analytical results for $r = a$ (= finite constant) and $r = \infty$, respectively. Furthermore, in this ring-shaped domain a series form of solution (equation (7)) which includes negative powers of r may be admitted. Consequently, though either Reiner's (1958) or Taylor & Saffman's (1957) physical explanations of this phenomenon and their assumption about the behaviour of the air may be correct, it is not impossible that a formal mathematical solution of equations (1)–(4) together with ($6a$) could be associated with a negative pressure gradient.

In the case considered here, the motion in the cylindrical region $1 > r, z < d/R$, like the flows considered by Batchelor and Stewartson, is governed by equations (1)–(4) and (6). However, the torsional motion is obtained in the present work as part of a solution in a wider domain, at the bounding surface of which conditions are imposed on all three velocity components. This analysis also differs from Stewartson's in that here a velocity field is assumed which is of a much more general nature than that given by equation (5). This is so, first, because in this work a series form is assumed and, secondly, because the coefficients in this series depend on all the boundary conditions of the problem. The proposed series form of solution could have been applied to the Batchelor–Stewartson problem merely by adding three boundary conditions at the surface $r = \infty$. However, arbitrary choices of additional conditions are unlikely to yield a solution which represents a flow that can be reproduced in a laboratory. The flow considered here was chosen as the subject of this work because it is bound to take place in a large, flat-bottomed tank when a long vertical cylindrical rod is rotated in it, and when this rotation is the only source of disturbance. Accordingly, the conditions imposed at the non-solid surfaces, $r > 1$, $z \to \infty$, and $z > 0$, $r \to \infty$ are that the three velocity components should be finite, while at the solid surfaces $z = d/R$, $r < 1$; $z > d/R$, $r = 1$; and $z = 0$ tangential and normal components of velocity are assumed to be known. Nevertheless, since the gap-to-diameter ratio of $1/8$ considered here is fairly small, it is possible to compare this analysis and Stewartson's, where this ratio is assumed to approach zero (though he compares his analytical results with experiments in which it is as large as $5/6$). The pressure distributions associated with the two solutions turn out to be radically different. Thus, in the present case, $\partial p/\partial r$ is not positive everywhere on the surface $z = d/R$, $r < 1$. Furthermore, unlike Stewartson's case, here the pressures at the stationary and rotating surface, $p(r, 0)$ and $p(r, d/R)$, are not equal. These results are consistent with the contention that Stewartson's flow is not the unique solution of the problem posed by him and renders questionable Popper & Reiner's interpretation of Stewartson's work.

2. Analytical development

In terms of the stream function $\psi(r, z)$ and the operator Δ defined by

$$u = r^{-1}(\partial\psi/\partial z), \quad w = -r^{-1}(\partial\psi/\partial r),$$

$$\Delta \equiv r\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\right\} + \frac{\partial^2}{\partial z^2},$$

equation (4) is satisfied identically, (2) reduces to

$$Re[(\partial\psi/\partial z)(\partial v/\partial r) - (\partial\psi/\partial r)(\partial v/\partial r) + r^{-1}v(\partial\psi/\partial z)] = \Delta(rv), \tag{2'}$$

and by cross differentiating (1) and (3) we obtain

$$Re\left\{\frac{1}{r}\left(\frac{\partial\psi}{\partial z}\right)\frac{\partial(\Delta\psi)}{\partial r} - \frac{1}{r}\left(\frac{\partial\psi}{\partial r}\right)\frac{\partial(\Delta\psi)}{\partial z} - \frac{2}{r^2}\left(\frac{\partial\psi}{\partial z}\right)\Delta\psi - 2v\frac{\partial v}{\partial z}\right\} = \Delta\Delta\psi. \tag{8}$$

As stated previously, the boundary conditions at the solid surfaces are

$$\left.\begin{aligned} v &= 0 \quad \text{at} \quad z = 0, \quad (0 < r < \infty), \\ &= r \quad \text{at} \quad z = d/R, \quad (0 < r < 1), \\ &= 1 \quad \text{at} \quad z > d/R \quad (r = 1), \end{aligned}\right\} \tag{9}$$

and

$$\psi = 0, \quad \partial\psi/\partial n = 0, \tag{10}$$

where n is the direction normal to these surfaces. The remaining boundary conditions are that

$$\left.\begin{aligned} &v, r^{-1}(\partial\psi/\partial z), \quad r^{-1}(\partial\psi/\partial r) \text{ be finite} \\ &\text{as} \quad r \to \infty \quad (z > 0), \quad \text{and as} \quad z \to \infty \quad (r > 1). \end{aligned}\right\} \tag{11}$$

It is assumed that v and ψ can be expressed in the forms

$$v = \sum_{j=0}^{\infty} V_{2j}(r,z) Re^{2j}, \quad \psi = \sum_{j=0}^{\infty} \Psi_{2j+1}(r,z) Re^{2j+1}. \tag{12}$$

The absence of odd and even powers of Re in the expressions for v and ψ, respectively, follows from the fact that v should, yet u and w should not, change sign with Ω. The expansions for v and ψ start with $(Re)^0$ and $(Re)^1$, respectively, because when Re vanishes equations (2′), (9) and (11) yield a non-trivial finite solution while (8) together with (10) and (11) yields a trivial one. These expressions are substituted into equations (2′) and (8) and then the coefficients of equal powers of Re are compared, yielding

$$0 = \Delta(rV_0), \tag{2_0}$$

$$-2V_0(\partial V_0/\partial z) = \Delta\Delta\Psi_1, \tag{8_1}$$

$$(\partial\Psi_1/\partial z)(\partial V_0/\partial r) - (\partial\Psi_1/\partial r)(\partial V_0/\partial z) + r^{-1}(\partial\Psi_1/\partial z)V_0 = \Delta(rV_2), \tag{2_2}$$

and so on. On the left-hand sides of equations (2_{2j}) and (8_{2j+1}) there are only coefficient functions of indices lower than $2j$ and $(2j+1)$, respectively. Hence by imposing the appropriate conditions on V_{2j} and Ψ_{2j+1}, we can obtain these functions as the solution of second- and fourth-order boundary-value problems. Though the boundary conditions (9), (10) and (11) can be satisfied in many different ways, we choose to let V_0 satisfy equations (9) and (11), make V_{2j}, Ψ_{2j-1} and $\partial\Psi_{2j-1}/\partial n$ for $j > 0$ finite at the non-solid boundaries $r > 1$, $z \to \infty$ and $z > 0$, $r \to \infty$ and make them vanish at the solid ones. The advantage of this choice is that the solution for each of the coefficients is independent of Re. Thus the solution obtained in this manner holds for a range $0 \leqslant Re < \hat{Re}$ where $\hat{Re}$ depends upon the convergence of the series expressions for v and ψ in the (r, z) domain under consideration.

When Re vanishes the solution has the following form:

$$u = w = 0, \quad v = V_0(r, z),$$

and since V_0 is independent of Re this is the so-called 'Stokes flow' for the problem considered.

As a first step towards the evaluation of V_0 we obtain a solution of the simplified boundary-value problem

$$\left.\begin{aligned} r^{-1}\Delta(rV_0) &= 0 \quad \text{for} \quad 1 < r < \infty, \quad 0 < z < \infty, \\ V_0(r,0) = 0, \quad V_0 &\text{ finite} \quad \text{as} \quad r, z \to \infty, \quad V_0(1,z) = 1. \end{aligned}\right\} \tag{13}$$

The first two boundary conditions together with the governing equation imply that V_0 can be expressed in the following Fourier-integral form

$$V_0 = \int_0^\infty [a(\gamma)\, K_1(\gamma r) + b(\gamma)\, I_1(\gamma r)] \sin(\gamma z)\, d\gamma, \tag{14}$$

in which I_1 and K_1 are the modified Bessel Functions of the first and second kind. In view of the finiteness of V_0 at infinity the former must be excluded, so that $b(\gamma)$ vanishes identically. The function $a(\gamma)$ is then obtained from the last of the boundary conditions to be satisfied. Thus the solution for $V_0(r,z)$ in the wider domain outside the gap and away from its inlet $d/R > z > 0$, $r = 1$, is given by

$$V_0 = \frac{2}{\pi}\int_0^\infty \frac{1}{\gamma}\frac{K_1(\gamma r)}{K_1(\gamma)} \sin(\gamma z)\, d\gamma. \tag{14'}$$

This expression decreases rapidly with r and approaches the value r^{-1} as z approaches infinity. These features will be shown to be compatible with results which follow from physical rather than mathematical considerations.

Inside the gap the solution given by (14′) does not hold, and at the vicinity of the inlet to the gap it is inaccurate. Hence a solution for V_0 which holds in the gap region has to be obtained otherwise. The latter was made to match the former solution along the boundaries $z = 4$, $1 < r < 5$ and $0 < z < 4$, $r = 5$; it satisfies equation (9) along the solid boundaries, and satisfies equation (2_0) everywhere in the domain bounded by these limits. This solution was obtained by means of the relaxation method described by Allen (1954). Thus instead of V_0 being evaluated as a continuous function of r and z it is evaluated at the nodal points of a rectangular grid of mesh lengths τ. We designate by $\chi_{(j)}$ the values of any continuous function χ at the nodal points $j = 0, 1, 2, \ldots, 12$ which are distributed around the local origin designated by O as shown in figure 1. It can be shown that the following relationship holds within an error of $O(\tau^4)$:

$$r^{-1}\Delta(r\chi) = \left\{\sum_{j=1}^{4} \chi_{(j)} + (2i)^{-1}(\chi_{(1)} - \chi_{(3)}) + (4 + i^{-2})\right\}\tau^{-2},$$

where $i = r/\tau$. Thus the values of V_0 are adjusted so that inside the (r,z)-domain considered, equation (2_0) is (approximately) satisfied at every one of the nodal points taken as a local origin, while at the nodal points on the boundary V_0 is given by (9). Figure 2 represents the numerical solution for $100V_0$ obtained with mesh length $\tau = \frac{1}{4}$. Figure 3 represents the solution in a smaller part of the gap area obtained by means of two consecutive refinements with mesh lengths $\tau = \frac{1}{8}$ and $\tau = \frac{1}{16}$. Boundary values in the maps of figures 2 and 3, which were calculated from equation (14′) or transferred from figure 2, are underlined. Other values of $100V_0$ along the 'internal' boundaries are obtained by interpolation.

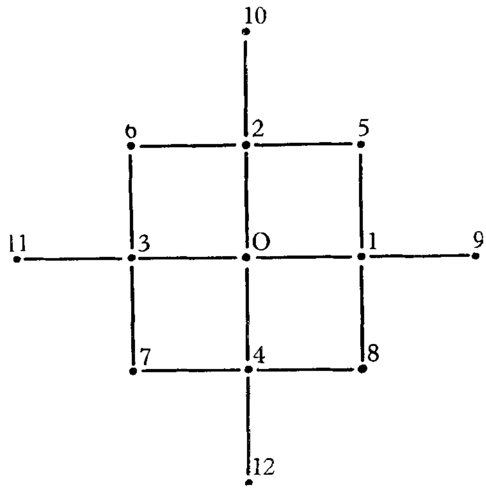

FIGURE 1. Scheme used in the relaxation procedure.

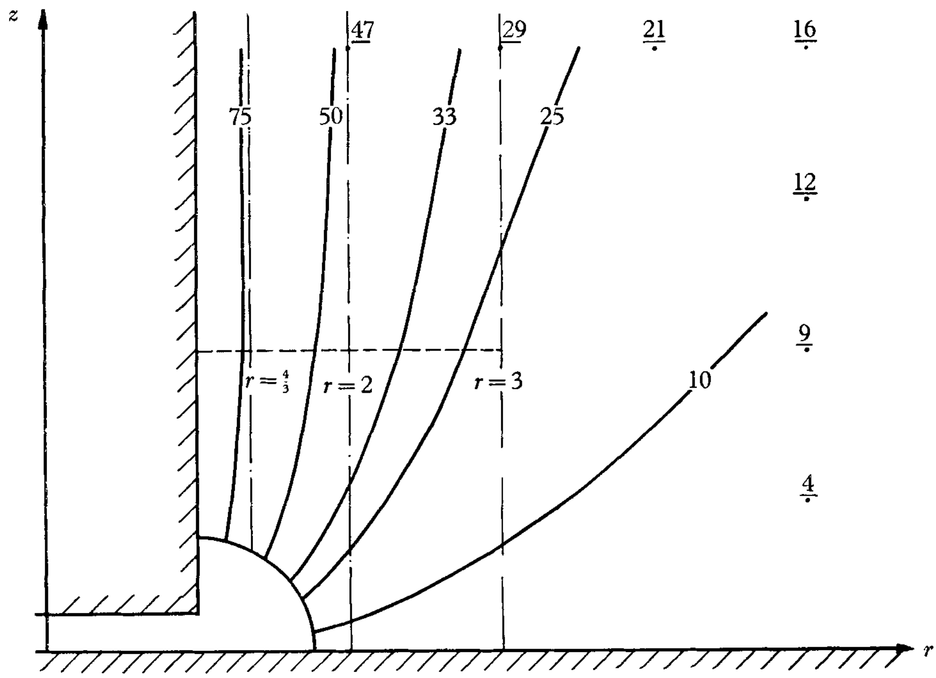

FIGURE 2. Solution for $100 V_0$ near the gap inlet.

Next Ψ_1' is obtained by making use of the solution for V_0 and the following finite difference approximations (see figure 1):

$$2\chi(\partial\chi/\partial z) = \chi_{(0)}(\chi_{(2)} - \chi_{(4)})/r,$$

$$\begin{aligned}\Delta\Delta\chi = r^{-4}\Big\{&20\chi_{(0)} - 8\sum_{k=1}^{4}\chi_{(k)} + 2\sum_{k=5}^{8}\chi_{(k)} + \sum_{k=9}^{12}\chi_{(k)} \\ &+ i^{-1}[4(\chi_{(1)} - \chi_{(3)}) + (\chi_{(11)} + \chi_{(6)} + \chi_{(7)}) - (\chi_{(9)} + \chi_{(5)} + \chi_{(8)}) \\ &+ 3i^{-2}[\chi_{(1)} + \chi_{(3)} - 2\chi_{(0)}] - \tfrac{3}{2}i^{-3}[\chi_{(1)} - \chi_{(3)}]\Big\}.\end{aligned}$$

The calculated values of $-64\times10^3\Psi_1$ are mapped in figures 4 and 5. These were obtained by first relaxing a comparatively wide net of mesh length $\tau=\frac{1}{4}$ and then refining the gap area by taking τ to be $\frac{1}{16}$. While the maximum residues (see Allen 1954, pp. 2–6) in the solution presented by figures 2 and 3 are around 1–2 units (i.e. 1 or 2 % of the maximum value of $|V_0|$), the residues in the solution for $-64\times10^3\Psi_1$ are as high as 30 units. However, the overall sum of residues in the latter is much smaller than the corresponding sum at the first stage of the relaxation procedure, when Ψ_1 was taken to be identically zero and the residues were the local values of $2V_0(\partial V_0/\partial z)r^4$. Consequently, though the solution for Ψ_1 is not by any means accurate, figures 4 and 5 represent a contribution to the flow pattern which is of the right order of magnitude and which is qualitatively correct.

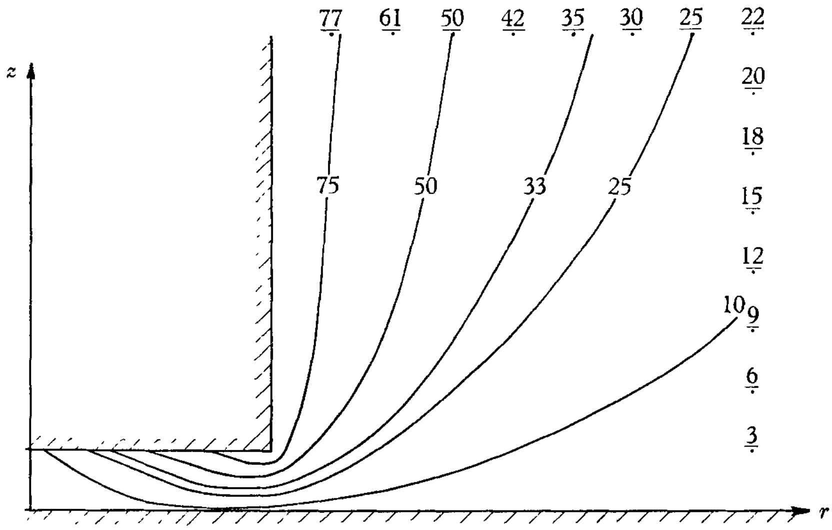

FIGURE 3. Solution for $100V_0$ at the inlet to and inside the gap.

In view of the asymptotic behaviour of V_0 (shown in figure 2), the initial residues $2V_0(\partial V_0/\partial z)$ drop very rapidly as $(r^2+z^2)^{\frac{1}{2}}$ increases. Since the solution mapped in figures 4 and 5 is obtained by varying the values of Ψ_1 in the vicinity of the points where the residues are non-zero or non-negligible, in the final stage of relaxation Ψ_1 also approaches zero as $(r^2+z^2)^{\frac{1}{2}}$ gets larger. This behaviour is not restricted to Ψ_1, and is in agreement with the more general treatment which is carried out in the Appendix. There, we obtain the solution of the boundary-value problems

$$\left.\begin{aligned} r^{-1}\Delta[r\Pi(r,z)] &= t(r,z),\\ \Pi(1,z)=0,\quad \Pi(r,0)=0,\quad &\Pi \text{ finite at } \infty,\end{aligned}\right\} \tag{15}$$

and

$$\left.\begin{aligned} \Delta\Delta\Gamma(r,z) &= q(r,z),\\ \Gamma=\partial\Gamma/\partial n=0 \quad \text{at} \quad r=1 \quad \text{and} \quad z=0, \quad &\Gamma,\partial\Gamma/\partial n \text{ finite as } \quad r,z\to\infty,\end{aligned}\right\} \tag{16}$$

in the simplified domain $\infty>r>1$, $\infty>z>0$. The functions $t(r,z)$ and $q(r,z)$ represent the left-hand sides of equations (2_{2j}) and (8_{2j+1}), respectively. It is shown that when these decrease sufficiently rapidly with (r^2+z^2), so also do the solutions Π and Γ, which designate any of the functions V_{2j} and Ψ_{2j+1}, respectively. Inasmuch as the presence of the gap does not essentially affect the solution in the wider field one may, from this study, infer information about the

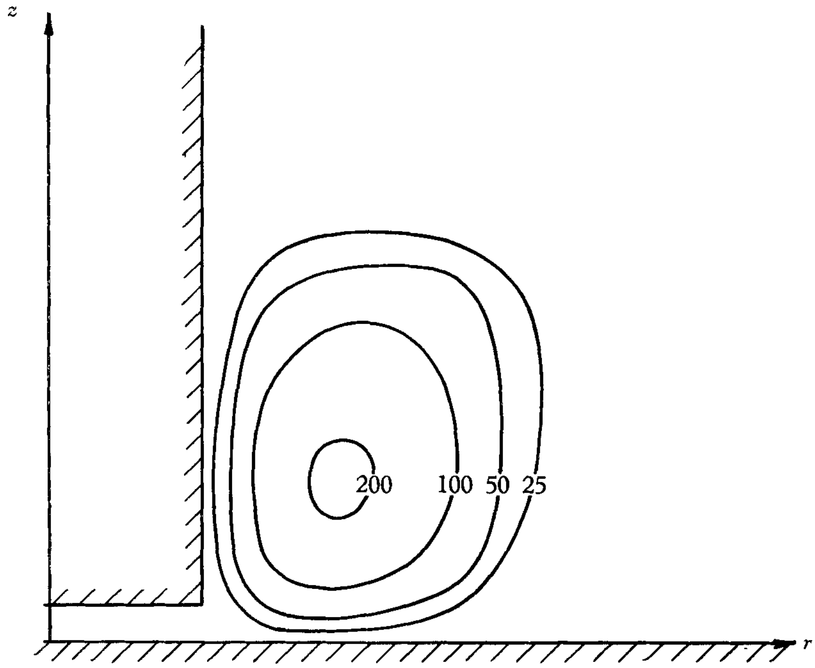

FIGURE 4. Solution for $-64\times 10^3\,\Psi_1$, near the gap inlet.

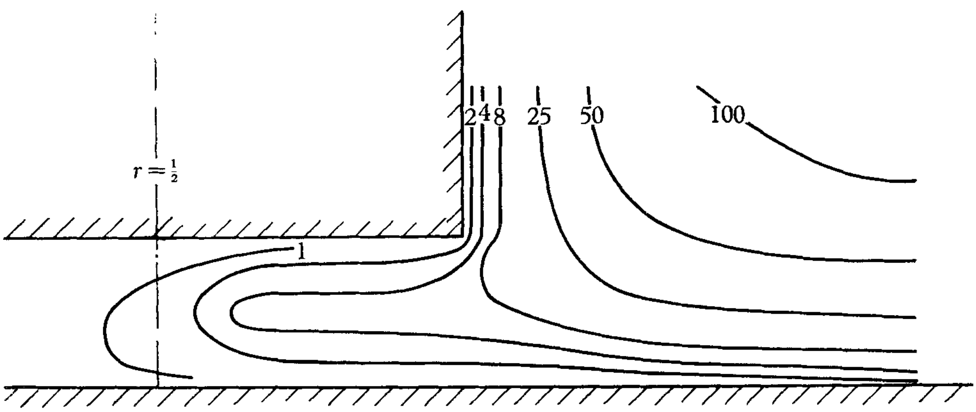

FIGURE 5. Solution for $-64\times 10^3\,\Psi_1$, at the inlet to and inside the gap.

asymptotic behaviour of V_{2j} and Ψ_{2j+1}. It is thus concluded that this feature of vanishing rapidly with increasing $(r^2+z^2)^{\frac{1}{2}}$ is carried from the V_{2j} and Ψ_{2j+1} of lower index to those of higher index. This conclusion leads to results which are physically plausible. Since, as z approaches infinity only V_0 is non-zero the flow there is given by

$$u = w = 0, \quad v = r^{-1}.$$

This is the flow which results from the steady rotation of an infinite, rather than semi-infinite, circular cylinder immersed in otherwise quiescent fluid. Indeed, this is the z-independent solution of equations (1)–(4) which satisfies the conditions of zero slip and zero normal velocity at the boundary $r = 1$, $-\infty < z < \infty$, and which is finite at infinity. Similarly, the vanishing of all the coefficient functions, and hence the entire flow, as r approaches infinity, is plausible, since it implies that the disturbance caused by the rotation of the cylinder is local.

3. Results and conclusions

When Re is 10 and only the first term is retained in each of the series (12) the maximum value of $(u^2+w^2)^{\frac{1}{2}}$ is about 5 % of the maximum value of $|v|$, which is unity. Furthermore, the absolute value of the left-hand side of equation (2_2) exceeds 5×10^{-3} only at a few nodal points near the inlet to the gap. The sign of this varies. Analysis similar to that carried out in the Appendix as well as experience in relaxation techniques with the differential operator of equation (2) thus indicates that when $Re = 10$, $|V_2 Re^2|$ may be as much as $\frac{1}{4}|V_0|$ but only in a small region. These relative orders of magnitude suggest the existence of a range of Reynolds numbers, perhaps $Re < 5$ roughly, in which the expressions for v and ψ converge at every point in the field. A rigorous proof of convergence may, perhaps, be achieved by obtaining a general expression for V_{2j} and Ψ_{2j+1}, in terms of the Green's functions for equations (2_{2j}) and (8_{2j+1}) similar to those obtained in the Appendix. It is significant that if the expressions for v and ψ do, in fact, converge, the solution for each of the coefficient functions and hence the solution for the flow is unique.

On the assumption that the first-term approximation with small Re is valid, inside the gap the circumferential component of velocity is found to be given by

$$v \simeq V_0 \simeq 4rz, \tag{17}$$

while outside the gap grad v is directed along concave curves which intersect the lines $r = 1$ and $z = 0$ at right angles. These curves are longer than d/R so that grad v is much bigger inside than outside the gap. Hence there is a region of abrupt transition near the inlet to the gap. The flow in the $\theta =$ const. planes can thus be treated as one which results from these variations of v^2, or of the centrifugal force associated with it. Thus the signs and locations of the streamlines $\psi = \Psi_1(Re)^1 =$ const. show that the motion in this plane is clockwise along closed paths. The paths which correspond to small $(-\psi)$ lie partly inside the gap, but the centre of this ring-vortex is near $r = 2$, $z = 1$.

Flow past a body is often treated as a combination of a 'main' infinite-Re potential flow and a boundary-layer 'correction'. The proposed analogous approach is applicable when the dominant effect is that of viscosity rather than that of inertia. Hence in the present analysis the zero-Re Stokes flow plays the role of the 'main' one and the motion in the $\theta =$ const. planes that of the 'correction'.

The pressure distribution along the surfaces $z = 0$ and $z = d/R$ is evaluated by assuming, together with equation (12),

$$p = \sum_{j=0}^{\infty} P_{2j}(r,z)\, Re^{2j}. \tag{12'}$$

It is further assumed that, provided *Re* is small, the first term is a satisfactory approximation. Equations (1) and (10) yield the following relationship,

$$r(\partial P_0/\partial r) = V_0^2 + \partial^3\Psi_1/\partial z^3.$$

Hence using equations (10) and (17) and the finite difference approximation to (8_1) and also using

$$\partial^3\chi/\partial z^3 = \{\chi_{(10)} - \chi_{(12)} + 2(\chi_{(4)} - \chi_{(2)})\}\, 2^{-1}\tau^{-3},$$

it is found that the pressure gradients are given approximately by

$$\frac{\partial P_0}{\partial r} = r + \left\{\frac{1}{r\tau^3}[4(\Psi_1)_{(4)} - (\Psi_1)_{(12)}] - 4\tau r\right\} \quad \text{at} \quad z = d/R,$$

and

$$\frac{\partial P_0}{\partial r} = \left\{\frac{1}{r\tau^3}[(\Psi_1)_{(10)} - 4(\Psi_1)_{(2)}]\right\} \quad \text{at} \quad z = 0.$$

In these relationships the expressions inside the braces are the contribution of the viscous forces to equation (1). Thus, according to the results presented in figure 5, the pressure gradient along $z = 0$ is positive and balances the difference between radial shear exerted by the solid surface and the smaller viscous shear acting in the $(-r)$ direction along an adjacent plane $z =$ const. Conversely, on $z = d/R$ the contribution of the viscous forces is negative and at some points near $r = \frac{3}{4}$ it is as big as the contribution of the centrifugal force, represented by the separate term r. This apparently contradicts the assertion that the flow in the (r, z)-plane is 'secondary' or one which follows the variations in centrifugal forces caused by gradients in the predominant 'primary' flow. However, it should be borne in mind that the 'secondary' flow inside the gap is 'driven' not only by the local variations in v^2 but also by those existing outside the gap.

Appendix. The vanishing of Γ and Π at infinity

It follows from equation (15) that the function $\eta(r,z)$ defined by

$$\partial\eta/\partial r = \Pi$$

is governed by the following differential system

$$\left.\begin{aligned} \nabla^2\eta &= -\int_r^\infty t(\hat{r}, z)\, d\hat{r}, \\ \eta(r,0) = 0, \quad \partial\eta/\partial r &= 0 \quad \text{at} \quad r = 1, \quad \eta \text{ finite at } \infty. \end{aligned}\right\} \tag{A 1}$$

Here ∇^2 is the usual three-dimensional Laplacian operator, so that the general solution of this problem can be obtained by making use of potential theory. Thus a Green's function, of the form

$$\begin{aligned} G_\eta(r,z,r',z') = &-(4\pi)^{-1}\int_{-\pi}^{\pi} \{[r^2 + r'^2 + 2rr'\cos\theta' + (z-z')^2]^{-\frac{1}{2}} \\ &- [r^2 + r'^2 + 2rr'\cos\theta' + (z+z')^2]^{-\frac{1}{2}}\}\, d\theta' \\ &+ \int_0^\infty c(\gamma)\, K_0(\gamma r)\sin(\gamma z)\, d\gamma, \end{aligned} \tag{A 2}$$

is sought in which the two bracketed terms of the first integral play the roles of point 'charges' or 'sources', located at the points $(r', \theta', -z')$ and $(r', \theta', +z')$.

The second integral is a rotationally symmetric function which is zero at the surface $z = 0$ and is harmonic throughout the space $r > 1$, $z > 0$. In view of the finiteness of G_η at infinity, equation (A 2), like (14), does not contain a term of the form $I_0(\gamma r)\sin(\gamma z)$. The function $c(\gamma)$ is evaluated by imposing the condition at $r = 1$. The Green's function sought is thus

$$\begin{aligned} G_\eta = -(4\pi)^{-1}\int_{-\pi}^{\pi}\Big\{&[r^2+r'^2+2rr'\cos\theta'+(z-z')^2]^{-\frac{1}{2}} \\ &-[r^2+r'^2+2rr'\cos\theta'+(z+z')^2]^{-\frac{1}{2}} \\ &-\frac{4}{\pi}\int_0^\infty \frac{(1+r'\cos\theta')K_1(\gamma\bar{r})K_0(\gamma r)}{\bar{r}K_1(\gamma)}\sin(\gamma z')\sin(\gamma z)\,d\gamma\Big\}\,d\theta', \end{aligned} \qquad \text{(A 2')}$$

where $\bar{r}$ is $(1+r'^2+2r'\cos\theta')^{\frac{1}{2}}$. Therefore, provided $t(r,z)$ at infinity is $O(r^2+z^2)^{-\mu}$ where $\mu > 1$, the function

$$\Pi = \int_0^\infty\int_1^\infty \frac{\partial G_\eta}{\partial r}(r,z,r',z')\left[-\int_{r'}^\infty t(\hat{r},z')\,d\hat{r}\right]r'\,dr'\,dz' \qquad \text{(A 3)}$$

has the desired asymptotic behaviour.

This conclusion is, of course, based on the assumption that the expression (A 2) for G_η contains a harmonic function which has a particular form. It should be stressed, however, that the choice of an infinite sine-transform only implies that, in accordance with the conditions of the problem, G_η is finite as z approaches infinity. Nevertheless, this choice, like that given by (14), does not necessarily imply that G_η must vanish at infinity. Consequently, though the foregoing is not a rigorous mathematical proof, one can safely assume that these results are valid.

It follows from equation (16) that the function ϕ defined by means of

$$r(\partial\phi/\partial r) \equiv \Gamma(r,z)$$

should be finite and have finite derivatives at infinity and also satisfy the following equations

$$\left.\begin{aligned} \nabla^2\nabla^2\phi = -\int_r^\infty \frac{q(\hat{r},z)}{\hat{r}}\,d\hat{r}, \\ \partial\phi/\partial r = \partial^2\phi/\partial r^2 = 0 \quad \text{at} \quad r = 1, \quad \phi = \partial\phi/\partial z = 0 \quad \text{at} \quad z = 0. \end{aligned}\right\} \qquad \text{(A 4)}$$

Again we seek a Green's function of the form

$$\begin{aligned} G_\phi = -(8\pi)^{-1}\Big\{\int_{-\pi}^{\pi}&\{[r^2+r'^2+2rr'\cos\theta'+(z-z')^2]^{\frac{1}{2}} \\ &-[r^2+r'^2+2rr'\cos\theta'+(z+z')^2]^{\frac{1}{2}}\}\,d\theta' + 4\pi zz'\int_0^\infty e^{-(z+z')\lambda}J_0(r'\lambda)\,J_0(r\lambda)\,d\lambda\Big\} \\ &+\int_0^\infty[m(\beta)K_0(\beta r)\sin(\beta z)+l(\beta)rK_1(\beta r)\sin(\beta z)+s(\beta)K_0(\beta r)z\cos(\beta z)]\,d\beta. \end{aligned} \qquad \text{(A 5)}$$

Since in terms of Cartesian co-ordinates x, y, z we have

$$\nabla^2(x^2+y^2+z^2)^{\frac{1}{2}} = 2(x^2+y^2+z^2)^{-\frac{1}{2}},$$

the two bracketed terms in the first integral are biharmonic everywhere, except at $(r', \theta', \pm z')$ where they have residues (in the potential theory rather than the relaxation or complex variable sense) of -8π. The integrals with respect to β and λ are either harmonic or biharmonic everywhere in the field. Again the requirement of finiteness of $\partial\phi/\partial r$ at infinity excludes from the infinite integrals terms of the form $rI_1(\beta r)\sin(\beta z)$, $I_0(\beta r)\sin(\beta z)$ and $ze^{\lambda z}J_0(\lambda r)$. In view of the z-wise skew-symmetry, $G_\phi(r,0,r',z')$ vanishes. By means of integration by parts, the three other boundary conditions together with the recurrence relationship $K_1(\beta r) = -K_0'(\beta r)$ yield

$$\int_0^\infty \left\{\beta m(\beta) + s(\beta) + \frac{d\{\beta l(\beta)\}}{d\beta}\right\} K_0(\beta r)\, d\beta - [\beta l(\beta) K_0(\beta r)]_0^\infty = 0,$$

$$\int_0^\infty \left\{-\beta m(\beta) K_1(\beta) - \beta l(\beta) K_0(\beta) + \frac{d\{\beta s(\beta) K_1(\beta)\}}{d\beta}\right\} \sin(\beta z)\, d\beta$$
$$- [\beta s(\beta) K_1(\beta)\sin(\beta z)]_0^\infty = \frac{1}{(8\pi)}(\partial H/\partial r)_{r=1},$$

$$\int_0^\infty \left\{-\beta^2 m(\beta) K_1'(\beta) - \beta l(\beta)[K_0(\beta) - \beta K_1(\beta)] + \frac{d\{\beta^2 s(\beta) K_1'(\beta)\}}{d\beta}\right\} \sin(\beta z)\, d\beta$$
$$- [\beta^2 s(\beta) K_1'(\beta)\sin(\beta z)]_0^\infty = \frac{1}{(8\pi)}\left(\frac{\partial^2 H}{\partial r^2}\right)_{r=1}. \quad \text{(A 6)}$$

Since it is assumed that

$$\lim_{\beta\to 0} s(\beta), \quad l(\beta), \quad m(\beta) = 0, \quad \text{(A 7)}$$

the boundary terms resulting from the integration by parts vanish. In equation (A 6) $H(r,z)$ is the sum of the first two integrals on the right-hand side of (A 5). Bateman (1959) has shown that the contributions of these integrals to the expression for $(\partial H/\partial z)_{z=0}$ represent the potential in the plane $z=0$ due to a uniform ring of radius r' situated in the plane $z = z'$ or $z = -z'$. These contributions are equal in magnitude but have opposite signs so that $\partial(H/\partial z)_{z=0}$ vanishes. Consequently the inverse of the sine- and Hankel-transforms of (A 6) yield

$$\beta m(\beta) + s(\beta) + d\{\beta l(\beta)\}/d\beta = 0,$$

$$-\beta m(\beta) K_1(\beta) - \beta l(\beta) K_0(\beta) + d\{\beta s(\beta) K_1(\beta)\}/d\beta$$
$$= \frac{1}{2\pi^2}\left\{\sin(\beta z')\int_{-\pi}^{\pi}(1 + r'\cos\theta')K_0(\bar{r}\beta)\, d\theta'\right.$$
$$\left. - 4\pi z'\int_0^\infty (\lambda^2+\beta^2)^{-2}\lambda^2\beta e^{-\lambda z'} J_0(r'\lambda) J_1(\lambda)\, d\lambda\right\},$$

$$-\beta^2 m(\beta) K_1'(\beta) - \beta l(\beta)[K_0(\beta) - \beta K_1(\beta)] + d\{\beta^2 s(\beta) K_1'(\beta)\}/d\beta$$
$$= \frac{1}{(2\pi^2)^2}\left\{\sin(\beta z')\int_{-\pi}^{\pi}(K_0(\bar{r}\beta) - \bar{r}^{-1}\beta(1 + r'\cos\theta')K_1(\bar{r}\beta))\, d\theta'\right.$$
$$\left. - 4\pi z'\int_0^\infty (\lambda^2+\beta^2)^{-2}\beta\lambda^3 e^{-\lambda z'} J_0(r'\lambda) J_1'(\lambda)\, d\lambda\right\}. \quad \text{(A 8)}$$

Without actually solving them it is possible to verify that these equations together with (A 7) yield unique solutions for $s(\beta)$, $l(\beta)$ and $m(\beta)$ and that when β is large these are $O(e^{\beta}\beta^{-\frac{5}{2}})$. It therefore follows that as r increases $\partial G_\phi/\partial r$ and its derivatives rapidly approach zero, and $\partial G_\phi/\partial r$ becomes independent of z as it approaches infinity. Thus, since away from the point (r', z') G_ϕ is biharmonic, and since $\partial G_\phi/\partial r$ and $\partial^2 G/\partial r^2$ vanish both for $r = 1$ and when r approaches infinity, $\partial G_\phi/\partial r$ is zero for large z, and hence everywhere at infinity.

Most of this work was carried out at the Aeronautics Department of the Imperial College, London. Mr S. Ladani and Mr M. Dinur of the Mechanical Engineering Department of the Technion in Haifa (Israel) computed the results given in figures 2–5. The author wishes to thank Mr P. G. Simpkins of the Aeronautics Department of the Imperial College for very useful comments and suggestions as well as for his help in preparing this paper.

REFERENCES

Allen, D. N. de G. 1954 *Relaxation Methods*. New York: McGraw-Hill.

Batchelor, G. K. 1951 *Quart. J. Math. Appl. Mech.* **4**, 29.

Bateman, H. 1959 *Partial Differential Equations of Mathematical Physics*, §7.41. Cambridge University Press.

Cobb, E. C. & Saunders, O. A. 1956 *Proc. Roy. Soc.* A, **236**, 343.

Cochrane, W. G. 1934 *Proc. Camb. Phil. Soc.* **30**, 365.

Kármán, T. von 1921 *Z. angew. Math. Mech.* **1**, 237.

Popper, B. A. & Reiner, M. 1956 *Brit. J. Appl. Phys.* **7**, 452.

Reiner, M. 1958 *Proc. Roy. Soc.* A, **247**, 152.

Stewartson, K. 1953 *Proc. Camb. Phil. Soc.* **49**, 333.

Taylor, Sir Geoffrey & Saffman, P. G. 1957 *J. Aero. Sci.* **21**, 553.

Wagner, C. 1948 *J. Appl. Phys.* **19**, 837.